1 Sheet 1

1.1 Excercise 1

- The matrix Lie group

$$SL_n = \{A \in \mathrm{Mat}_{n\times n} \mid \det(A) = 1\}.$$

  is a closed submanifold of the ambient space $\mathrm{Mat}_{n\times n}$ of codimension 1. Then, the tangent space $\mathrm{Lie}(SL_n) = T_1 SL_n$ is a linear subspace of $T_1 \mathrm{Mat}_{n\times n} \cong \mathrm{Mat}_{n\times n}$ of codimension 1. The linear condition specifying such subspace is given by the position

$$X \in \mathrm{Lie}(SL_n) \iff 1 + tX \in SL_n \text{ for } t \text{ small enough.} \tag{1}$$

  Condition (1) tells us that

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} \det(1+tX) \\ &= \lim_{t\to 0} \frac{1}{t}\left(\det(1+tX) - 1\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left(t^n(t^{-n} + t^{-n+1}\operatorname{tr}(X) + \dots + \det(X)) - 1\right) = \operatorname{tr}(X). \end{aligned}$$

- Consider the Lie group

$$O_n = \{A \in \mathrm{Mat}_{n\times n} \mid A^T A = 1\},$$

  We plug (1) in the definition of O_n and look at the first-order term

$$1 = (1+tX)(1+tX^T) = 1 + t(X + X^T) + \dots$$

  We conclude that $\mathrm{Lie(O_n)} = \{X \in \mathrm{Mat}_{n\times n} \mid X + X^T = 0\}$.

- Let $\Omega = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in \mathrm{Mat}_{2n\times 2n}$, and define

$$Sp_{2n} = \{A \in \mathrm{Mat}_{2n\times 2n} \mid A^T \Omega A = \Omega\}. \tag{2}$$

  Proceeding as above, we find

$$\mathrm{Lie}(Sp_{2n}) = \{X \in \mathrm{Mat}_{2n\times 2n} \mid \Omega X + X^T \Omega = 0\}.$$

- Let the unitary group be

$$U(n) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \overline{A}^T A = 1\}.$$

  The above procedure yields

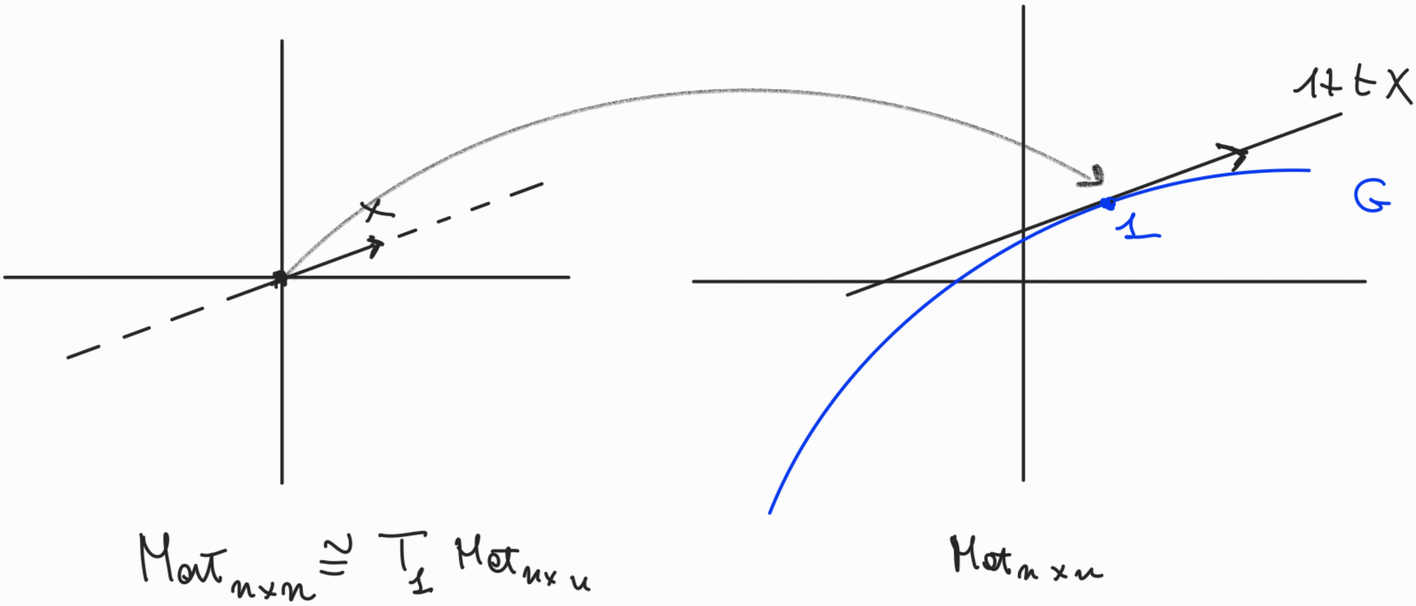
$$\mathrm{Lie}(U(n)) = \{X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \overline{X}^T + X = 0\}.$$

- Finally, we have the special unitary group

$$SU(n) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \overline{A}^T A = 1 \text{ and } \det(A) = 1\} = U(n) \cap SL_{2n}.$$

  Then it follows immediatly that

$$\mathrm{Lie}(SU(n)) = \{X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \overline{X}^T + X = 0 \text{ and } \operatorname{tr}(X) = 0\}.$$



Linear approximation of

$$\mathrm{Lie}\, G = \{ X \in \mathrm{Mat}_{n \times n} \mid e^{tX} \in G \ \forall t \in X \}.$$

1.2 Excercise 2

For a discussion on convergence and properties of the matrix exponential, see [2].
We observe that if $XX' = X'X$, then $\exp(X)\exp(X') = \exp(X + X')$. In particular

$$\exp((t+t')X) = \exp(\mathrm{tX})\exp(t'X). \tag{3}$$

1.3 Excercise 3

We keep track of the terms up to order 2 in t, t':

$$\begin{aligned}
&\exp(\mathrm{tX})\exp(t'X')\exp(\mathrm{tX})^{-1}\exp(\mathrm{tX'})^{-1} \\
&= \left(1 + tX + \frac{t^2X^2}{2} + \dots\right)\left(1 + t'X' + \frac{t'^2X'^2}{2} + \dots\right)\exp(-tX)\exp(-t'X') \\
&= \left(1 + t'X' + \frac{t'^2X'^2}{2} + tX + tt'XX' + \frac{t^2X^2}{2}\dots\right)\left(1 - tX + \frac{t^2X^2}{2} + \dots\right)\left(1 - t'X' + \frac{t'^2X'^2}{2} + \dots\right) \\
&= \left(1 + t'X' + \frac{t'^2X'^2}{2} + tt'[X, X'] + \dots\right)\left(1 - t'X' + \frac{t'^2X'^2}{2} + \dots\right) \\
&= 1 + tt'[X, X'] + \dots
\end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{d}{dt}|_{t=0}\frac{d}{dt'}|_{t'=0}\exp(\mathrm{tX})\exp(t'X')\exp(\mathrm{tX})^{-1} \\
&= \frac{d}{dt}|_{t=0}\frac{d}{dt'}|_{t'=0}\left(1 + t'X' + \frac{t'^2X'^2}{2} + tt'[X, X'] + \dots\right) = [X, X'].
\end{aligned} \tag{4}$$

1.4 Excercise 4

For further discussion concerning excercises 4 and 5, look at [1].

Let M be a smooth manifold and let

$$\Phi \colon \mathbb{R} \times M \to M$$

be a C^∞ group action. Denote as $\Phi_t \colon M \to M$ the restriction to the second factor, and denote as $\Phi_p \colon \mathbb{R} \to M$ the restriction to the first factor. Then we can define a smooth vector field on M by taking the velocity of the curve Φ_t at each point p:

$$X_p f = \frac{d}{dt}|_{t=0}(\Phi(t,p)) := \Phi_{p*}\left(\frac{d}{dt}|_{t=0}\right) f = \lim_{t\to 0}\frac{f(\Phi_t(p)) - f(p)}{t}. \tag{5}$$

Then X is the velocity field of the action Φ.
Viceversa, if we are given a velocity field X on M, we can try and look for a flow Φ such that (5) holds, that is, we are trying to integrate the vector field X. For every point $p \in M$, this amounts to solving the differential equation

$$\begin{cases} \frac{d}{dt}(\Phi(t,p)) = X_{\Phi(t,p)} \\ \Phi(0,p) = p. \end{cases} \tag{6}$$

Standard results from analysis give us the following.

Theorem 1. *The data of M and X uniquely determine an open subset*

$$W = \{(t,p) \in \mathbb{R} \times M \mid \alpha(p) < t < \beta(p)\} \subset \mathbb{R} \times M, \tag{7}$$

and a map $\Phi \colon W \to M$ such that the following holds.

1. *$\{0\} \times M \subset W$ and $\Phi(0,p) = p$ for all $p \in M$.*

2. *For all fixed $p \in M$, let $\Phi_p(t) = \Phi(t,p)$. Then*

$$\Phi_p \colon (\alpha(p), \beta(p)) \to M \tag{8}$$

is a C^∞ maximal soluiton of (6)*.*

3. *For each $p \in M$ there is a neighborhood V and a $\delta > 0$ such that $(-\delta, \delta) \times V \subset W$ and the restriction $\Phi|_V$ to $(-\delta,\delta) \times V$ is a C^∞ solution of*

$$\begin{cases} \dfrac{d}{dt}(\Phi|_V(t,q)) = X_{\Phi|_V(t,q)} \\ \Phi|_V(0,q) = q & \text{for all } q \in V \end{cases}.$$

We call W the local flow *associated to X.*

Slightly more precisely, we can rewrite (6) by taking local coordinates for M around the point p. This yields an autonomous system of differential equation for the components of Φ. Then, the first two points follow (essentially) from the theorem of existence and uniquenes of ODEs. Moreover we observe that the differential equation (6) depends smoothly also on the initial condition. This allows us to apply a stronger form of the theorem of existence and uniqueness, which yields (essentially) the third point. It also follows from uniqueness that

$$\Phi(t, \Phi(s,p)) = \Phi(t+s, p)$$

as long as both sides make sense.
The nontrivial part of the above theorem is that W is open in $\mathbb{R} \times M$. In general, the equality $W = \mathbb{R} \times M$ does not hold, since the open interval $(\alpha(p), \beta(p))$ might get arbitrarily small as p "goes to infinity". However, there are cases in which the equality holds, such as when M is compact or when $M = G$ is a Lie group and X is left invariant. In these cases we say that the flow Φ is global. The following lemma is easy to verify.

Lemma 1. *Let X be a vector field on the Lie group G. The following are equivalent.*

1. *X is left-invariant*

2. *The flow $\Phi \colon W \to G$ of X is left-invariant, i.e. $h\Phi(t,g) = \Phi(t,hg)$ for all t,h,g.*

Using this lemma, we can finally solve our excercise.

Theorem 2. *Let G be a Lie group and let X be a left inveriant vector field. Then, for every $h \in G$ there exists a unique group homomorphism*

$$\gamma \colon \mathbb{R} \to G \tag{9}$$

satisfying

$$\begin{cases} \dfrac{d}{dt}\gamma(t) = X_{\gamma(t)} \\ \gamma(0) = h. \end{cases} \tag{10}$$

Proof. By the third point in the Theorem, there exists a neighborhood V of the unit $e \in G$ and a positive number $\delta > 0$ such that the local flow $\Phi(t,g)$ of X is defined in $(-\delta,\delta) \times G$. Let $h \in G$ be any point. By the above Lemma, we see that the flow Φ is defined in $(-\delta,\delta) \times hV$. We conclude that for every $h \in G$ there is a neighborhood $U = hV$ such that $(-\delta,\delta) \times U$ is contained in W, and δ is constant. This easily implies that

$$W = \mathbb{R} \times G.$$

By uniqueness, each flow line

$$\Phi_h \colon \mathbb{R} \to G$$

is a group homomorphism and verifies the requests. □

Finally if we denote as $\Phi^X : \mathbb{R} \times G \to G$ the (global) action determined by the left invariant vector field X, we set

$$\exp(X) = \Phi^X(1, e).$$

It is easy to see that

1. $\exp(tX) = \Phi^X(t, e)$.
2. $\exp(-tX) = \Phi^X(t, e)^{-1}$.

1.5 Excercise 5

We need another form for the Lie bracket of Excercise 4. Let $\Phi^X : \mathbb{R} \times G \to G$ be the flow of the left invariant vector field X, and denote as $\Phi^X_t : G \to G$ the restriction to the second factor. Then we have the equality of tangent vectors

$$[X, X']_e = \frac{d}{dt}|_{t=0} \Phi^X_{-t,*}(X'_{\Phi^X_t(e)}). \tag{11}$$

The right hand side in (11) is the derivative of the function

$$Z : \mathbb{R} \to T_e G \cong \mathbb{R}^n, t \mapsto \Phi^X_{-t,*}(X'_{\Phi^X_t(e)}), \tag{12}$$

which can be recast by using Lemma 1 as follows. Applying the definition of pushforward, we get

$$Z(t) = \frac{d}{dt'}|_{t'=0} \Phi^X(-t, \Phi^{X'}(t', \Phi^X(t, e))) = \frac{d}{dt'}|_{t'=0} \Phi^X(t, e)\Phi^{X'}(t', e)\Phi^X(-t, e).$$

Thus

$$[X, X']_e = \frac{d}{dt}|_{t=0} \frac{d}{dt'}|_{t'=0} \Phi^X(t, e)\Phi^{X'}(t', e)\Phi^X(-t, e). \tag{13}$$

This form of the bracket will be useful in the second excercise sheet.

We can also show that (in analogy with the case of matrices)

$$[X, X']_e = \frac{d}{dt}|_{t=0} \frac{d}{dt'}|_{t'=0} \Phi^X(t, e)\Phi^{X'}(t', e)\Phi^X(-t, e)\Phi^{X'}(-t', e). \tag{14}$$

In fact, we use the Lemma 1 to write

$$\Phi^X(t, e)\Phi^{X'}(t', e)\Phi^X(-t, e)\Phi^{X'}(-t', e) = \Phi^{X'}(t', \Phi^X(t, e))\Phi^{X'}(-t', \Phi^X(-t, e)), \tag{15}$$

and apply the following "chain rule".

Lemma 2. *The derivative of a produttori of one parameter groups is*

$$\frac{d}{dt}|_{t'=0} \rho(t)\eta(t) = L_{\rho(0)*}\eta'(0) + R_{\eta(0)*}\rho'(0), \tag{16}$$

where L_h and R_h are the diffeomorphisms of left and right multiplication respectively.

Then we have

$$\begin{aligned} &\frac{d}{dt}|_{t'=0} \Phi^{X'}(t', \Phi^X(t, e))\Phi^{X'}(-t', \Phi^X(-t, e)) \\ &= R_{\Phi^X(-t,e),*} X'_{\Phi^X(t,e)} - L_{\Phi^X(t,e),*} X'_{\Phi^X(-t,e)} \\ &= R_{\Phi^X(-t,e),*} X'_{\Phi^X(t,e)} - X_e \end{aligned}$$

Finally, if f is the germ of a smooth function at e,

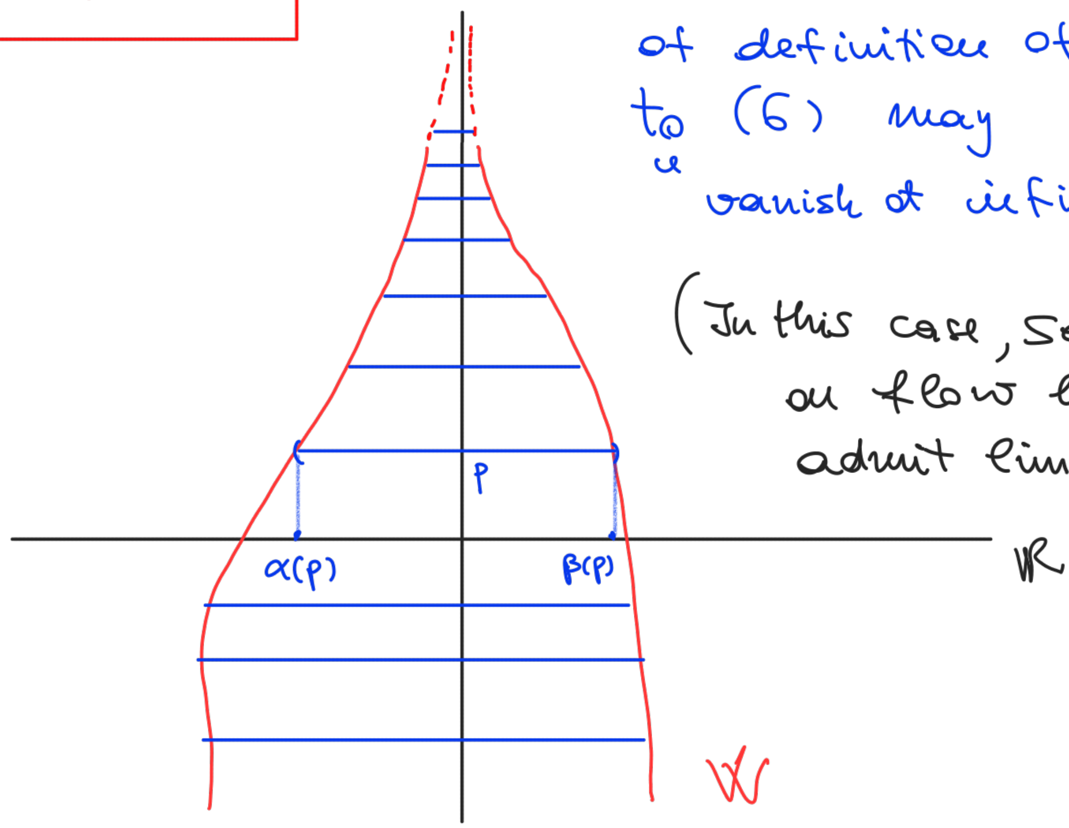
$$\begin{aligned}
R_{\Phi^X(-t,e),*}X'_{\Phi^X(t,e)}(f) &= X'_{\Phi^X(t,e)}(f\Phi^X(-t,e)) \\
&= X'_{\Phi^X(t,e)}(\Phi^X(-t,f)) \\
&= \Phi^X_{-t,*}X'_{\Phi^X(t,e)}(f)
\end{aligned}$$

where we used Lemma 1 again. Derivating again at $t = 0$ and using (13) we get our claim.

EXERCISE 4

In the maximal interval of definition of the solution to (6) may "vanish at infinity".

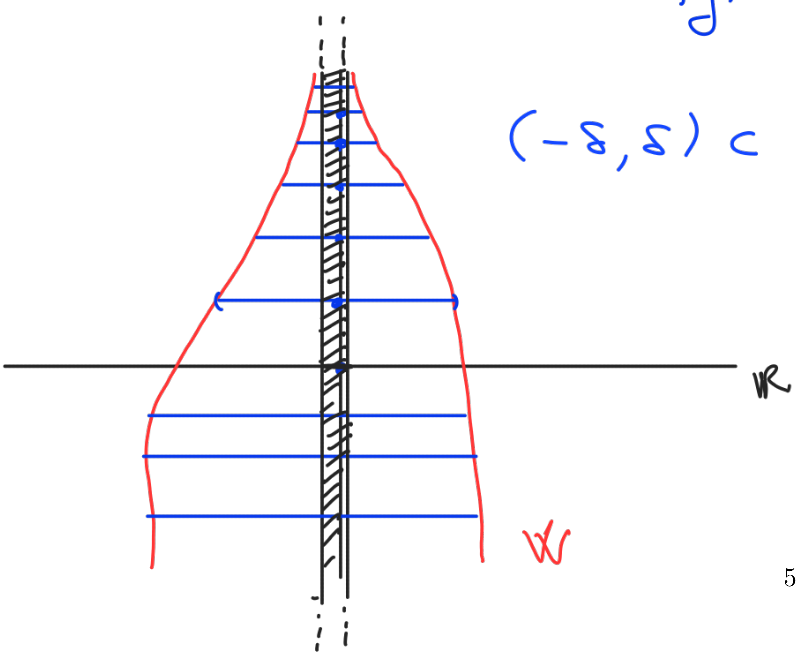
(In this case, sequences of points on flow lines do not admit limits)



If $M = G$ is a Lie group and X is left invariant,

$$h\Phi^X(t,g) = \Phi^X(t,hg).$$

$(-\delta,\delta) \subset W \Rightarrow$ By uniqueness, it is possible to extend flow lines to $\mathbb{R}$, i.e. $W = \mathbb{R} \times M$.



References

[1] W. M. Boothby. An introduction to differentiable manifolds and riemannian geometry. 1975.

[2] Brian C. Hall. Lie groups, lie algebras, and representations: An elementary introduction. 2004.